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A NONLINEAR TAUBERIAN THEOREM IN FUNCTION THEORY[1]

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1. Introduction and summary.

Let

$$f(z) = \prod_{k=1}^{\infty} \left(1+\frac{z}{a_k}\right) \qquad (0 < a_k \leqq a_{k+1}) \tag{1.1}$$

be an entire function of genus zero with real negative zeros. We denote by

$$\lambda = \lambda(f) = \limsup_{r\to\infty} \frac{\log\log f(r)}{\log r}$$

the *order* of such an f, and by $n(r)=n(r, 0; f)$ the number of zeros in the disk $|z| \leqq r$.

Valiron [14] discovered the following connection between the growth of a function (1.1) and the distribution of its zeros:

$$n(r) \sim Cr^{\lambda} \text{ iff } \log f(r) \sim C\pi(\csc \pi\lambda)r^{\lambda} \qquad (r \to \infty) \tag{1.2}$$

where C is a constant and $0<\lambda<1$. The well-known formula

$$\log f(z) = \int_0^{\infty} \frac{zn(t)}{t+z}\frac{dt}{t} \qquad (|\arg z| < \pi) \tag{1.3}$$

exhibits $\log f(r)$ as a linear integral transform of n, so that (1.2) can be viewed as a linear abelian-tauberian theorem of a kind frequently studied.

The main purpose of this paper is to prove an analogue of Valiron's theorem in which $\log f(r)$ is replaced by $T(r)=T(r,f)$, the Nevanlinna characteristic of f. Edrei and Fuchs [2] have shown that, for functions of the form (1.1), T admits the representation

$$T(r) = \sup_{0<\theta<\pi} \int_0^{\infty} P\left(\frac{r}{t}, \theta\right) N(t)t^{-1}\, dt \tag{1.4}$$

where

$$N(r) = \int_0^r n(t)t^{-1}\, dt, \qquad P(t, \theta) = \frac{1}{\pi}\frac{\sin\theta}{t+t^{-1}+2\cos\theta}$$

so that T is a nonlinear integral transform of N.

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Using (1.4), it is not difficult to prove the following abelian result [4, Corollary 2.1]:

Let f be a function of the form (1.1) *and let L be a slowly varying function. Then*

$$N(r) \sim r^{\lambda}L(r) \qquad (r \to \infty, 0 < \lambda < 1)$$

implies

$$(1.5) \qquad \begin{aligned} T(r) &\sim r^{\lambda}L(r) && \text{if } 0 < \lambda \leqq \tfrac{1}{2}, \\ &\sim (\csc \pi\lambda) r^{\lambda}L(r) && \text{if } \tfrac{1}{2} < \lambda < 1. \end{aligned}$$

The term "slowly varying" here is used in the sense of Karamata [10] and means that L is positive and satisfies

$$\lim_{r\to\infty} \frac{L(\sigma r)}{L(r)} = 1$$

for every $\sigma > 0$.

Our theorem is the tauberian converse of the statement above.

THEOREM 1. *Let f be an entire function of genus zero and let L be a slowly varying function. Suppose that* $T(r) \sim r^{\lambda}L(r)$ $(r \to \infty)$.

(a) *If* $0 \leqq \lambda \leqq \frac{1}{2}$, *then* $N(r) \sim r^{\lambda}L(r)$.

(b) *If* $\frac{1}{2} \leqq \lambda \leqq 1$ *and if f has the form* (1.1), *then* $N(r) \sim (\sin \pi\lambda) r^{\lambda}L(r)$.

Valiron's theorem is an easy consequence of Wiener's general tauberian theorem [11, p. 78]. Our proof of Theorem 1, part (b), first uses potential theory to "linearize", the problem, then we invoke Wiener's theorem. As far as we know, this is the first application of the general tauberian theorem to a transform of the "sup" type (1.4).

The author had originally proved part (a) of Theorem 1 only for functions of the form (1.1). He thanks the referee for pointing out that the hypothesis of negative zeros is unnecessary, and for indicating the simple proof that appears in §4.

When Theorem 1 is combined with several known tauberian results for entire functions, we obtain a long chain of statements concerning asymptotic behavior of the functions (1.1). Let us say that the function ϕ *varies regularly with order* λ if $\phi(r) \sim r^{\lambda}L(r)$ $(r \to \infty)$ for some slowly varying function L; then we have

COROLLARY 1. *If f has the form* (1.1), *then regular variation of order* λ $(0 < \lambda < 1)$ *of any one of the functionals* $n(r,f)$, $N(r,f)$, $T(r,f)$, $\log f(r)$, $A(r,f)$ *implies regular variation* (*of order* λ) *of all of them, and also the existence of a limit* (*as* $r \to \infty$), *which depends only on* λ, *for the ratio of any two of them.*

Here the quantity $A(r,f)$ denotes, as usual, the functional defined by

$$A(r,f) = \frac{1}{\pi} \int_{|z| \leqq r} \left(\frac{|f'(z)|}{1+|f(z)|^2} \right)^2 dx\, dy \qquad (z = x+iy).$$

Part of Corollary 1 remains true when $\lambda = 0$ or $\lambda = 1$; we examine this situation in §7.

That regular variation is about the "right" kind of asymptotic behavior to consider in the present context is suggested by the existence of "ratio" tauberian theorems converse to Corollary 1. As one example, Edrei and Fuchs [4], [5] have shown that the existence of $\lim_{r\to\infty} N(r)/T(r)$ when $\frac{1}{2}<\lambda\leqq 1$ implies regular variation of T. Other results along this line have been obtained by Karamata [10], Drasin [1], and Shea [12].

The abelian theorem (1.5) and the Edrei-Fuchs ratio theorem in fact hold for the class of meromorphic functions with negative zeros and positive poles. This, together with some examples, leads one to conjecture that if f is such a function, then regular variation of T implies regular variation of both $N(r, 0)$ and $N(r, \infty)$, where $N(r, 0)=N(r)$ as used above, and $N(r, \infty)$ is the analogous counting function for the poles of f. The integral representation becomes

$$T(r) = \sup_{0<\theta<\pi} \int_0^\infty [P(r/t, \theta)N(t, 0)+P(r/t, \pi-\theta)N(t, \infty)]t^{-1}\, dt$$

so that an affirmative answer to the conjecture would perhaps constitute a unique sort of tauberian theorem. It does not appear possible to adapt our proof of Theorem 1 to this more complicated problem.

2. **Lemmas for convolutions and slowly varying functions.** Our proofs make use of the notations

$$\phi \# \Psi(r) = \int_0^\infty \phi\left(\frac{r}{t}\right)\Psi(t)t^{-1}\, dt \qquad (0 < r < \infty), \tag{2.1}$$

$$\phi_1 * \Psi_1(x) = \int_{-\infty}^\infty \phi_1(x-y)\Psi_1(y)\, dy \qquad (-\infty < x < \infty), \tag{2.2}$$

where the integrals are assumed to exist in the Lebesgue sense. Note that, when $\phi_1(x)=\phi(e^x)$ and $\Psi_1(x)=\Psi(e^x)$, then $\phi \# \Psi(e^x)=\phi_1 * \Psi_1(x)$.

The following well-known facts about convolutions on $(0, \infty)$ will be used in the sequel: if H and K are both integrable on $(0, \infty)$ with respect to $t^{-1}\, dt$, then so is $H \# K$. Further, the Mellin transform

$$\hat{K}(x) = \int_0^\infty K(t)t^{-ix}t^{-1}\, dt$$

satisfies

$$(H \# K)\hat{}(x) = \hat{H}(x)\hat{K}(x) \qquad (-\infty < x < \infty).$$

We shall use the general tauberian theorem in the following form [7, p. 286].

Wiener's Theorem. *Suppose that*

$$g \text{ is real}, \qquad g \in L^\infty(0, \infty), \quad \text{and} \quad \int_0^\infty |K(t)|t^{-1}\, dt < \infty.$$

Then

$$\lim_{r\to\infty} g \# K(r) = C \quad \textit{implies} \quad \lim_{r\to\infty} g(r) = C\left(\int_0^\infty K(t)t^{-1}\,dt\right)^{-1}$$

provided

$$\text{(SD)} \qquad \liminf_{r\to\infty;\sigma\to 1+} g(\sigma r)-g(r) \geqq 0$$

and $\hat{K}(x)\neq 0$ $(-\infty<x<\infty)$.

Explicitly, condition (SD) means that, given $\varepsilon>0$, there exists $\sigma_0>1$ and r_0 such that $g(\sigma r)-g(r)>-\varepsilon$ holds whenever $1\leqq\sigma\leqq\sigma_0$ and $r>r_0$. We shall say that such functions are *slowly decreasing*.

The following "cancellation lemma" appears to be new. It often provides a convenient tool for "eliminating" the slowly varying term $L(r)$ in the proof of an abelian or tauberian theorem.

LEMMA 1. *Suppose that L varies slowly, $g\in L^\infty(0,\infty)$, and K satisfies*

$$\text{(2.3)} \qquad \int_0^1 |K(t)|t^{-a}t^{-1}\,dt+\int_1^\infty |K(t)|t^{a}t^{-1}\,dt < \infty$$

for some $a>0$. Then

$$\text{(2.4)} \qquad (Lg)\# K(r)/L(r) = g\# K(r)+o(1) \qquad (r\to\infty).$$

Here the term Lg denotes pointwise multiplication. The existence of $(Lg)\# K$ follows from (2.3), as we shall show below. We have also implicitly assumed that L is measurable, but since any slowly varying function can be written

$$L(r) = c(r)\exp\left(\int_1^r \varepsilon(t)t^{-1}\,dt\right)$$

where

$$\text{(2.5)} \qquad \lim_{r\to\infty} c(r) = c > 0 \quad\text{and}\quad \lim_{r\to\infty}\varepsilon(r) = 0$$

[10, p. 45], and since all our main results are about asymptotic equivalence at $+\infty$, we may as well assume that *all our slowly varying functions are of the form*

$$\text{(2.6)} \qquad L(r) = c\exp\left(\int_1^r \varepsilon(t)t^{-1}\,dt\right),$$

where $\varepsilon(t)$ is bounded and measurable on $(0,\infty)$, vanishes for $t<1$, and satisfies (2.5).

From (2.6) it is clear that $L(\sigma r)/L(r)\to 1$ $(r\to\infty)$ uniformly for $\sigma\in[a,b]$ $(0<a<b<\infty)$. In our proofs of Lemma 1 and Theorem 1 we shall need the following slightly stronger statement concerning uniform convergence.

LEMMA 2. *If L is a slowly varying function, there exists a positive increasing sequence r_n such that*

$$\lim_{n\to\infty} \frac{r_{n+1}}{r_n} = \infty$$

and such that

$$\lim_{n\to\infty} \sup_{r_n \leqq r \leqq r_{n+1}} \left| \log \frac{L(r)}{L(r_n)} \right| = 0.$$

Proof. $L(r)$ has the form (2.6), and the lemma is trivial if $\varepsilon(t)=0$ for all sufficiently large t. Otherwise,

$$\delta(x) = \sup_{t \geqq x} |\varepsilon(e^t)|$$

is strictly positive and nonincreasing for all x, and we may set

$$\Delta(x) = \int_0^x \delta(t)^{-1/2}\, dt \qquad (x > 0).$$

Put $x_n = \Delta(n)$ $(n=1, 2, \ldots)$, and notice that

$$x_{n+1} - x_n = \int_n^{n+1} \delta(t)^{-1/2}\, dt \geqq \delta(n)^{-1/2} \to \infty \qquad (n \to \infty),$$

while

$$\int_{x_n}^{x_{n+1}} \delta(t)\, dt = \int_n^{n+1} \delta(\Delta(x))\delta^{-1/2}(x)\, dx \leqq \int_n^{n+1} \delta(x)^{1/2}\, dx = o(1) \qquad (n \to \infty).$$

The lemma follows from the choice $r_n = \exp(x_n)$.

Proof of Lemma 1. We first prove the existence of $(Lg) \# K$. Set

$$\phi(x) = L(e^x) = c \exp\left(\int_0^x \eta(t)\, dt\right),$$

where by (2.6) $\eta(t) = \varepsilon(e^t)$ is in $L^\infty(-\infty, \infty)$, vanishes for $t<0$, and tends to 0 at $+\infty$.

Let $\gamma > 0$ be given, and choose x_1 such that $|\eta(t)| < \gamma/2$ $(t > x_1)$. Then

$$\left| \log \frac{\phi(t)}{\phi(x)} \right| \leqq \left| \int_x^t |\eta(t)|\, dt \right| \leqq \frac{\gamma}{2} |x-t|$$

whenever $x_1 \leqq x \leqq t$ or $x_1 \leqq t \leqq x$, and similarly

$$\left| \log \frac{\phi(t)}{\phi(x)} \right| \leqq \frac{\gamma}{2}(x - x_1) + \int_0^{x_1} |\eta(t)|\, dt \qquad (t \leqq x_1 \leqq x).$$

If we choose $x_0 > x_1$ such that

$$\int_0^{x_1} |\eta(t)|\, dt < \frac{\gamma}{2}(x - x_1) \qquad (x > x_0)$$

we deduce that

$$\phi(t) \leqq \phi(x) e^{\gamma|x-t|} \tag{2.7}$$

holds for all $t \in (-\infty, \infty)$ when $x \geqq x_0$. Now set

$$Q(x) = K(e^x), \qquad h(x) = g(e^x), \qquad A(x, t) = \phi(t)/\phi(x) - 1.$$

The hypothesis (2.3) becomes

$$\int_{-\infty}^{\infty} e^{a|t|}|Q(t)|\,dt < \infty,$$

and so, by (2.7), there exists x_0 such that

$$\lim_{T\to\infty} \int_{-\infty}^{-T} + \int_{T}^{\infty} |A(x, x-t)Q(t)|\,dt = 0 \tag{2.8}$$

uniformly for $x_0 \leqq x < \infty$.

Choose, by Lemma 2, an increasing sequence x_n such that

$$x_{n+1} - x_n \to \infty \qquad (n \to \infty) \tag{2.9}$$

and such that

$$\sup |A(x, t)| \to 0 \qquad (n \to \infty) \tag{2.10}$$

where the sup is taken over all $x \in [x_n, x_{n+1}]$ and $t \in [x_{n-1}, x_{n+2}]$.

Then, for $x_n \leqq x \leqq x_{n+1}$,

$$\begin{aligned}\frac{(\phi h) * Q(x)}{\phi(x)} &= h * Q(x) + \int_{-\infty}^{\infty} A(x, t)h(t)Q(x-t)\,dt \\ &= h * Q(x) + \int_{x-x_{n-1}}^{\infty} + \int_{-\infty}^{x_{n+2}-x} A(x, x-t)h(x-t)Q(t)\,dt \\ &\quad + \int_{x_{n-1}}^{x_{n+2}} A(x, t)h(t)Q(x-t)\,dt.\end{aligned}$$

Since $h \in L^\infty$, the two middle integrals are $o(1)$ $(n \to \infty)$ because of (2.9) and (2.8), and the last integral is likewise because of (2.10) and the condition $Q \in L^1$.

Hence $((\phi h) * Q)/\phi = g * Q + o(1)$ at $+\infty$, and this is equivalent to (2.4).

In our convolutions, the "bounded factor" g will often be of a special form. The following asserts that such functions satisfy the hypothesis (SD) of Wiener's theorem; its proof is an immediate consequence of the definition.

LEMMA 3. *Suppose $g(t) = G(t)/t^\lambda L(t) \in L^\infty(0, \infty)$ with G nondecreasing, L slowly varying, and $\lambda \geqq 0$. Then g is slowly decreasing.*

It is convenient to have on hand the following little abelian theorem, whose proof may easily be supplied by the reader.

LEMMA 4. *Suppose $g \in L^\infty(0, \infty)$ and $\int_0^\infty |K(t)|t^{-1}\,dt < \infty$. Then*

(i) $\lim_{r\to\infty} g(r) = a$ *implies* $\lim_{r\to\infty} g \# K(r) = a\int_0^\infty K(t)t^{-1}\,dt$.

(ii) *if g is real and $K \geqq 0$, then*

$$\limsup_{r\to\infty} g \# K(r) \leqq \left(\limsup_{r\to\infty} g(r)\right) \int_0^\infty K(t)t^{-1}\,dt$$

and

$$\liminf_{r\to\infty} g \# K(r) \geqq \left(\liminf_{r\to\infty} g(r)\right) \int_0^\infty K(t)t^{-1}\,dt.$$

3. **Preliminary results on $T(r)$.** Throughout this section, f is assumed to have the form (1.1). Define

$$F(z) = \frac{1}{\pi}\int_0^z \log f(\zeta)\frac{d\zeta}{\zeta} \qquad (|\arg z| < \pi) \tag{3.1}$$

where the integration is taken over any path which avoids the negative real axis. In particular, if $z=re^{i\theta}$, with $0<\theta<\pi$, we can integrate along the real axis from 0 to r, then along $|\zeta|=r$, and use the fact that $\log f(x)$ is real to get

$$\operatorname{Im} F(re^{i\theta}) = \frac{1}{\pi}\int_0^\theta \log|f(re^{it})|\,dt \qquad (0 \leqq \theta < \pi). \tag{3.2}$$

From (3.1) and Valiron's formula (1.3), we deduce that

$$F(z) = \frac{1}{\pi}\int_0^\infty \frac{zN(t)}{t+z}\frac{dt}{t} \qquad (|\arg z| < \pi). \tag{3.3}$$

Taking imaginary parts in (3.3) yields

$$\operatorname{Im} F(re^{i\theta}) = \frac{1}{\pi}\int_0^\infty \frac{r\sin\theta\, N(t)}{t^2+r^2+2tr\cos\theta}\,d\theta, \tag{3.4}$$

from which it follows that

$$r^{-\lambda}\operatorname{Im} F(re^{i\theta}) = (t^{-\lambda}N) \# K_{\lambda,\theta}(r) \qquad (0 < \theta < \pi), \tag{3.5}$$

where

$$K_{\lambda,\theta}(t) = \frac{\sin\theta}{\pi}\frac{t^{-\lambda}}{t+t^{-1}+2\cos\theta}. \tag{3.6}$$

For a general meromorphic function g, the Nevanlinna characteristic is defined by

$$T(r,g) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \log^+|g(re^{i\theta})|\,d\theta + N(r,\infty;g), \tag{3.7}$$

where, when $g(0)\neq\infty$,

$$N(r,\infty;g) = \int_0^r \{\text{number of poles in } |z| < t\}t^{-1}\,dt$$

and $\log^+ x = \max(\log x, 0)$.

Now if f has the form (1.1), it is easy to show that, when r is fixed

(3.8) $\log|f(re^{i\theta})|$ *decreases as θ increases from* 0 *to* π.

From this, there follows the existence of a unique number $\beta=\beta(r)\in(0,\pi]$ such that, for $\theta\in[0,\pi]$,

$$|f(re^{i\theta})| \geqq 1 \quad \text{iff} \quad 0 \leqq \theta \leqq \beta. \tag{3.9}$$

Moreover, $|f(re^{i\theta})|$ is an even function of θ, so (3.9), (3.7), and (3.2) yield

$$T(r,f) = \operatorname{Im} F(re^{i\beta}) = \sup_{0<\theta<\pi} \operatorname{Im} F(re^{i\theta}). \tag{3.10}$$

(When $\beta(r)=\pi$, we have $|f(re^{i\pi})|\geqq 1$, thus $\operatorname{Im} F(re^{i\pi})$ can be defined by taking $\theta=\pi$ in (3.2).) When (3.10) and (3.4) are combined, we see that we have derived the Edrei-Fuchs formula (1.4) for T in terms of N.

Jensen's theorem, applied to functions (1.1), asserts that

$$N(r) = \frac{1}{\pi}\int_0^{\pi} \log |f(re^{i\theta})|\, d\theta.$$

Using this and various results above, we infer that

$$N(r) \leqq T(r) \leqq \log f(r), \tag{3.11}$$

$$T(r,f) = T(r, 1/f), \tag{3.12}$$

and

$$N(r) = \operatorname{Im} F(re^{i\pi}). \tag{3.13}$$

Finally, we shall require:

LEMMA OF EDREI-FUCHS [3, p. 322]. *If h is meromorphic and E is a set with Lebesgue measure $\mu(E)$, then*

$$\frac{1}{\pi}\int_E \log^+ |h(re^{i\theta})|\, d\theta \leqq \frac{11R}{R-r} T(R, h)\mu(E)\left[1+\log^+ \frac{1}{\mu(E)}\right] \qquad (1 < r < R).$$

4. **Proof of Theorem 1(a).** Let $\{r_n\}$ be any positive, increasing, unbounded sequence. Then the hypothesis $T(r)\sim r^{\lambda}L(r)$ implies that $\{r_n\}$ is a sequence of Pólya peaks for T, in the sense of [15, p. 82]. To see this, let $\delta(x)$ be as in the proof of Lemma 2, and define sequences $\{B_n\}$ and $\{b_n\}$ by

$$\log B_n = \int_{\log r_n}^{1+\log r_n} \delta(t)^{-1/2}\, dt, \qquad \log b_n = -\min\left(\int_{(\log r_n)-1}^{\log r_n} \delta(t)^{-1/2}\, dt, \tfrac{1}{2}\log r_n\right).$$

It is easily verified that

$$\lim_{n\to\infty} b_n = 0, \qquad \lim_{n\to\infty} B_n = \lim_{n\to\infty} b_n r_n = +\infty,$$

$$\limsup_{n\to\infty}\left\{\left|\log\frac{L(r)}{L(r_n)}\right| : b_n r_n \leqq r \leqq B_n r_n\right\} = 0.$$

The last statement leads to

$$T(r) \leqq (1+\gamma_n)(r/r_n)^{\lambda}T(r_n) \qquad (b_n r_n \leqq r \leqq B_n r_n)$$

where $\lim_{n\to\infty}\gamma_n=0$, which is the defining inequality for Pólya peaks.

Put $J(r)=\{\theta : |f(re^{i\theta})|<1\}$. If $\lambda>0$, then Theorem 2 of [15] (with $\tau=\infty$, $q=1$, $\mu=\lambda$) yields

$$\lim_{n\to\infty} \operatorname{meas} J(r_n) = 0. \tag{4.1}$$

When $\lambda=0$, a straightforward modification of the proof of Lemma 1 of [15] shows that (4.1) still holds.

From Jensen's formula, the definition (3.7) of $T(r)$, the Edrei-Fuchs Lemma, and (4.1), it follows that

$$0 \leqq T(r_n)-N(r_n) = \frac{1}{2\pi}\int_{J(r_n)} \log |f(r_n e^{i\theta})|^{-1}\, d\theta \leqq 11T\left(2r_n, \frac{1}{f}\right)\cdot \varepsilon_n, \tag{4.2}$$

with $\lim_{n\to\infty} \varepsilon_n=0$.

We can assume that $f(0)=1$, so that $T(r, 1/f)=T(r,f)=T(r)$. Moreover, from $T(r)\sim r^\lambda L(r)$ follows $T(2r)=O(T(r))$. So from (4.2) we deduce

$$\lim_{n\to\infty} \frac{N(r_n)}{T(r_n)} = 1.$$

The sequence $\{r_n\}$ was arbitrary, thus $N(r)\sim T(r)$, which is the desired conclusion

5. **Proof of Theorem 1(b).** Choose, by Lemma 2, an increasing sequence $\{r_n\}$ such that

$$\lim_{n\to\infty} \frac{r_{n+1}}{r_n} = \infty \tag{5.1}$$

and such that

$$\lim_{n\to\infty} \sup_{r_n\leqq r\leqq r_{n+3}} \left|\log \frac{L(r)}{L(r_n)}\right| = 0. \tag{5.2}$$

Define a sequence of "interrupted annuli" D_n by

$$\begin{aligned} D_n &= \{z : r_n < |z| < r_{n+3}, |\arg z| < \beta(r)\} \\ &= \{z : r_n < |z| < r_{n+3}, |f(z)| > 1\} \end{aligned} \tag{5.3}$$

(see §3), and define the harmonic function u_n on D_n by

$$u_n(z) = \log |F(z)/z^\lambda L(r_n)| \qquad (z\in D_n), \tag{5.4}$$

where F is defined by (3.1).

We claim:

There exist positive numbers m and n_0 such that

$$u_n(z) > -m \qquad (z\in D_n, n\geqq n_0). \tag{5.5}$$

Proof of (5.5). By (3.2) and the definition of β, we have

$$\operatorname{Im} F(re^{i\theta}) \geqq \operatorname{Im} F(re^{i\pi/4}) \qquad (\pi/4 \leqq \theta \leqq \beta(r)). \tag{5.6}$$

By (3.4) with $\theta=\pi/4$, Valiron's formula (1.3), and (3.11),

$$\operatorname{Im} F(re^{i\pi/4}) \geqq \frac{r}{\pi\sqrt{2}}\int_0^\infty \frac{N(t)}{(t+r)^2}\,dt = \frac{r}{\pi\sqrt{2}}\int_0^\infty \frac{n(t)}{t+r}\frac{dt}{t} = \frac{\log f(r)}{\pi\sqrt{2}} \geqq \frac{T(r)}{\pi\sqrt{2}}. \tag{5.7}$$

Combining (5.6) and (5.7), we have

$$(5.8)\qquad |F(re^{i\theta})| \geqq \operatorname{Im} F(re^{i\theta}) > \frac{T(r)}{\pi\sqrt{2}} \qquad (\pi/4 \leqq \theta \leqq \beta(r)).$$

To estimate $|F(z)|$ for z close to the positive axis, we first observe that

$$\frac{2(ts\cos\theta+t^2)}{t^2+s^2+2ts\cos\theta} \geqq \frac{t}{t+s} \qquad \left(s,\, t > 0,\, 0 \leqq \theta \leqq \frac{\pi}{4}\right).$$

Then, it follows from (3.1) with $\zeta = te^{i\theta}$ and from taking real parts in Valiron's formula (1.3) that

$$\operatorname{Re} F(re^{i\theta}) = \frac{1}{\pi}\int_0^r \log|f(te^{i\theta})|t^{-1}\,dt = \frac{1}{\pi}\int_0^r \frac{dt}{t}\int_0^\infty \frac{ts\cos\theta+t^2}{t^2+s^2+2ts\cos\theta}\,\frac{n(s)}{s}\,ds$$

$$\geqq \frac{1}{\pi\sqrt{2}}\int_0^r \frac{dt}{t}\int_0^\infty \frac{tn(s)}{t+s}\,\frac{ds}{s} = \frac{1}{\pi\sqrt{2}}\int_0^r \log f(t)\,\frac{dt}{t} \qquad \left(0 \leqq \theta \leqq \frac{\pi}{4}\right).$$

Thus, by the monotonicity of $\log f(t)$, and (3.11),

$$(5.9)\quad |F(re^{i\theta})| \geqq \operatorname{Re} F(re^{i\theta}) \geqq \log f(r/e)/\pi\sqrt{2} \geqq T(r/e)/\pi\sqrt{2} \qquad (0 \leqq \theta \leqq \pi/4).$$

Since T is increasing and $\overline{F(z)} = F(\bar z)$, (5.8) and (5.9) show that

$$(5.10)\qquad |F(re^{i\theta})| \geqq T(r/e)/\pi\sqrt{2} \qquad (|\theta| \leqq \beta(r),\, 0 < r < \infty).$$

From (5.10), (5.2), and the hypotheses $T(r)\sim r^\lambda L(r)$, $L(r/e)\sim L(r)$ we deduce that

$$|F(z)| > r^\lambda L(r_n)/2\pi e\sqrt{2} \qquad (z \in D_n,\, n \geqq n_0)$$

which completes the proof of (5.5).

The next step in the proof is to replace the "O" statement (5.5) by a "o" statement. We will call upon potential theory to show that

$$(5.11)\qquad \lim_{n\to\infty}\ \inf_{r_{n+1}\leqq r\leqq r_{n+2}} u_n(r) \geqq 0.$$

Proof of (5.11). The function

$$w = \phi(z) = (i/2)(z^{1/2}-z^{-1/2}) \qquad (|\arg z| < \pi)$$

maps the domain $D=\{z : |z|>1,\, -\pi<\arg z<\pi\}$ conformally onto the upper half plane $\operatorname{Im} w>0$, and maps the circle $|z|=1$ (with -1 deleted) onto the interval $-1<w<1$. As z tends to a point on $(-\infty, -1)$, $\phi(z)$ approaches the real axis outside $-1<w<1$. Thus

$$P(z) = \frac{1}{\pi}\arg\left[\frac{\phi(z)-1}{\phi(z)+1}\right] \qquad (0 < \arg < \pi)$$

is harmonic in D and satisfies

$$(5.12)\qquad \begin{gathered} 0 < P(z) < 1 \qquad (z \in D), \\ \lim_{z\to e^{i\theta}} P(z) = 1 \qquad (|\theta| < \pi), \qquad \lim_{z\to t} P(z) = 0 \qquad (t < -1). \end{gathered}$$

Moreover, the identity

$$\arg((it-1)/(it+1)) = 2\cot^{-1} t \qquad (t > 0, 0 < \arg < \pi, 0 < \cot^{-1} < \pi/2)$$

shows that

$$P(x) = (2/\pi)\cot^{-1}((x^{1/2}-x^{-1/2})/2) < 8x^{-1/2}/\pi \qquad (x > 16). \tag{5.13}$$

The boundary of each domain D_n defined by (5.3) consists of the four arcs

$$\{|z| = r_n, |\arg z| \leqq \beta(r_n)\}, \qquad \{|z| = r_{n+3}, |\arg z| \leqq \beta(r_{n+3})\},$$
$$\{re^{i\beta(r)} : r_n \leqq r \leqq r_{n+3}\}, \qquad \{re^{-i\beta(r)} : r_n \leqq r \leqq r_{n+3}\}.$$

We denote by $\tilde{\gamma}=\tilde{\gamma}(n)$ the union of the first two arcs, and by $\tilde{\beta}=\tilde{\beta}(n)$ the union of the latter two. (Since perhaps $\beta(r)=\pi$ for some r's, the arcs in $\tilde{\beta}$ may coincide in whole or in part, but this will not affect our arguments.) The sets $\tilde{\gamma}$ and $\tilde{\beta}$ intersect only at the four (or fewer) "corner points" $r_n \exp(\pm i\beta(r_n))$, $r_{n+3}\exp(\pm i\beta(r_{n+3}))$.

Putting

$$v_n(z) = P(z/r_n)+P(r_{n+3}/z) \qquad (r_n < |z| < r_{n+3}, |\arg z| < \pi),$$

we see that v_n is harmonic and positive on D_n, and it follows from (5.12) that $\lim v_n(z)$ exists as z tends to any noncorner point of the boundary. Also

$$\begin{aligned} &\lim_{z\to\zeta} v_n(z) > 1 \qquad (\zeta \in \tilde{\gamma}(n), \zeta \text{ not a corner}),\\ &\lim_{z\to\zeta} v_n(z) \geqq 0 \qquad (\zeta \in \tilde{\beta}(n), \zeta \text{ not a corner}). \end{aligned} \tag{5.14}$$

From the definition (5.4) and from (3.10), we have

$$u_n(r\exp(\pm i\beta(r))) \geqq \log\frac{\operatorname{Im} F(re^{i\beta})}{r^\lambda L(r_n)} = \log\frac{T(r)}{r^\lambda L(r)}+\log\frac{L(r_n)}{L(r)} \qquad (r_n \leqq r \leqq r_{n+3}).$$

So, if we fix $\varepsilon>0$, by (5.2) and the original hypothesis there exists n_1 such that

$$\liminf_{z\to\zeta} u_n(z) > -\varepsilon \qquad (\zeta \in \tilde{\beta}(n), n \geqq n_1). \tag{5.15}$$

We assume that n_1 is larger than the n_0 of (5.5). Then (5.5), (5.14) and (5.15) show that

$$\liminf_{z\to\zeta} mv_n(z)+u_n(z)+\varepsilon > 0 \qquad (n \geqq n_1)$$

for every noncorner boundary point ζ of D_n. In addition,

$$mv_n(z)+u_n(z)+\varepsilon > -m \qquad (z \in D_n, n \geqq n_1),$$

and the corner points form a set of capacity zero, so the maximum principle for harmonic functions [13, p. 77] yields

$$u_n(z) > -mv_n(z)-\varepsilon \qquad (z \in D_n, n \geqq n_1). \tag{5.16}$$

We can further assume that $r_{n+1}/r_n > 16$ $(n \geqq n_1)$. Then it follows from (5.13) that

$$v_n(r) < \frac{8}{\pi}\left[\left(\frac{r_{n+1}}{r_n}\right)^{-1/2} + \left(\frac{r_{n+3}}{r_{n+2}}\right)^{-1/2}\right] \qquad (r_{n+1} \leqq r \leqq r_{n+2},\ n \geqq n_1).$$

Putting the last inequality into (5.16) and using (5.1), we see that

$$\lim_{n\to\infty} \inf_{r_{n+1}\leqq r\leqq r_{n+2}} u_n(r) > -\varepsilon.$$

Since ε was arbitrary, (5.11) is established.

From (5.11), (5.2), and (5.4), we obtain the key inequality

$$\liminf_{r\to\infty} \frac{F(r)}{r^\lambda L(r)} \geqq 1. \tag{5.17}$$

The rest of the proof consists of preparations for and applications of Wiener's theorem.

Suppose first that $\frac{1}{2} < \lambda < 1$. Define $g(t) = N(t)/t^\lambda L(t)$, and put

$$K(t) = K_{\lambda,\pi/2\lambda}(t) = \frac{\sin(\pi/2\lambda)}{\pi}\,\frac{t^{-\lambda}}{t+t^{-1}+2\cos(\pi/2\lambda)}.$$

Then, by (3.5) and (3.10),

$$(Lg) \# K(r) = r^{-\lambda}\,\mathrm{Im}\, F(re^{i\pi/2\lambda}) \leqq r^{-\lambda}T(r). \tag{K}$$

Lemma 1, together with the above inequality, yields

$$\limsup_{r\to\infty} g \# K(r) = \limsup_{r\to\infty} \frac{(Lg) \# K(r)}{L(r)} \leqq \limsup_{r\to\infty} \frac{T(r)}{r^\lambda L(r)} = 1. \tag{5.18}$$

To obtain a complementary lim inf inequality, we start with

$$F(r) = \frac{1}{\pi}\int_0^\infty \frac{rN(t)}{t+r}\,\frac{dt}{t},$$

which is formula (3.3). Dividing both sides by r^λ, this becomes

$$r^{-\lambda}F(r) = (Lg) \# H(r) \tag{H}$$

with $H(t) = t^{1-\lambda}/\pi(1+t)$.

Using Lemma 1, together with (5.17), we find that

$$\liminf_{r\to\infty} g \# H(r) = \liminf_{r\to\infty} \frac{(Lg) \# H(r)}{L(r)} = \liminf_{r\to\infty} \frac{F(r)}{r^\lambda L(r)} \geqq 1. \tag{5.19}$$

Applying Lemma 4 to (5.18) and (5.19) gives

$$\limsup_{r\to\infty} (g \# K) \# H(r) \leqq \int_0^\infty H(t)t^{-1}\,dt,$$

$$\liminf_{r\to\infty} (g \# H) \# K(r) \geqq \int_0^\infty K(t)t^{-1}\,dt.$$

Both integrals on the right are equal to $\csc \pi\lambda$, and convolution is commutative and associative, so

$$\lim_{r\to\infty} g \# (H \# K)(r) = \csc \pi\lambda. \tag{5.20}$$

Since

$$(H \# K)\hat{}(x) = \hat{H}(x)\hat{K}(x)$$

with

$$\hat{H}(x) = \csc \pi(\lambda+ix) \neq 0, \qquad \hat{K}(x) = \sin\left[\frac{\pi}{2\lambda}(\lambda+ix)\right] \csc \pi(\lambda+ix) \neq 0$$
$$(-\infty < x < \infty),$$

and g is slowly decreasing (by Lemma 3), Wiener's theorem applied to (5.20) gives

$$\lim_{r\to\infty} g(r) = \frac{\csc \pi\lambda}{(\csc \pi\lambda)^2} = \sin \pi\lambda,$$

and this completes the proof of the theorem when $\lambda < 1$.

When $\lambda = 1$, the representations (H) and (K) are still valid, but the above argument breaks down because the kernels H and K fail to satisfy the requisite integrability conditions. To remedy the situation, we integrate by parts. Put

$$Q(r) = \int_r^\infty \frac{N(t)}{t^2}\,dt, \qquad K_1(t) = \frac{2t^2}{\pi(t^2+1)^2}, \qquad H_1(t) = \frac{t}{\pi(1+t)^2}.$$

It is easily checked that

$$r^{-1}F(r) = Q \# H_1(r) = (L(Q/L)) \# H_1(r), \tag{H$_1$}$$

$$r^{-1} \operatorname{Im} F(re^{i\pi/2}) = Q \# K_1(r) = (L(Q/L)) \# K_1(r) \tag{K$_1$}$$

and that

$$\hat{K}_1(x) = \frac{x \sin (\pi/2)(1+ix)}{i \sin \pi(1+ix)} \neq 0, \qquad \hat{H}_1(x) = \frac{x}{\sinh \pi x} \neq 0.$$

Assuming for the moment that Q/L is bounded and that $-Q/L$ is slowly decreasing, we deduce as before that

$$\lim_{r\to\infty} Q/L(r) = \pi. \tag{5.21}$$

Thus Q varies slowly, and so we have, as $r \to \infty$

$$\frac{Q(2r)}{Q(r)} = 1 - \left[\int_r^{2r} \frac{N(t)}{t^2}\,dt\right] \Big/ Q(r) \to 1.$$

This implies that, as $r \to \infty$,

$$\frac{N(r)}{2rQ(r)} \leq \left[\int_r^{2r} \frac{N(t)}{t^2}\,dt\right] \Big/ Q(r) = o(1)$$

which, together with (5.21), shows that $N(r) = o(rL(r))$, as required.

To prove the boundedness of Q/L, we first observe that

$$Q(r) \leqq 4\int_r^\infty \left(\frac{t}{t+r}\right)^2 \frac{N(t)}{t^2}\,dt \leqq \frac{4}{r}\int_0^\infty \frac{rN(t)}{(t+r)^2}\,dt = \frac{4\log f(r)}{r}.$$

Nevanlinna's inequality for entire functions (1.1) yields ([8, p. 18])

$$\log f(r) \leqq 3T(2r).$$

Thus

$$Q(r) = O(T(2r)/r) = O(L(2r)) = O(L(r)) \qquad (r \to \infty).$$

That $-Q/L$ is slowly decreasing follows from Lemma 3 and the fact that Q is decreasing.

6. **Proof of Corollary 1.** Throughout this section, "$\sim$" denotes asymptotic equivalence as the independent variable tends to infinity, and L denotes a slowly varying function.

Consider the diagram

$$\begin{tikzcd}
 & n \arrow[dl, shift left, "2"'] \arrow[dr, shift left] \arrow[dd, "3"] & \\
N \arrow[ur, shift left] & & \log f \arrow[ul, shift left, "1"'] \\
 & T \arrow[ul, "4"] \arrow[r, leftrightarrow, "5"'] & A
\end{tikzcd}$$

in which an arrow from X to Y is read "if X varies regularly, then so does Y". We shall verify the existence of the implications (1)–(5).

(1) That

$$n(r) \sim r^\lambda L(r) \text{ iff } \log f(r) \sim \pi(\csc \pi\lambda) r^\lambda L(r)$$

follows from the original Valiron theorem [14, p. 237] and the connection between proximate orders and slowly varying functions. (See [9, p. 27] for a discussion of this.)

(2) For a proof of the elementary abelian result

$$n(r) \sim r^\lambda L(r) \quad \text{implies } N(r) \sim r^\lambda \lambda^{-1} L(r)$$

see [6, p. 273].

Conversely, the classical method of Landau can be used [4, p. 26] to show that

$$N(r) \sim r^\lambda L(r) \quad \text{implies } n(r) \sim \lambda r^\lambda L(r).$$

(3) The nonlinear abelian result

$$\begin{aligned} n(r) \sim r^\lambda L(r) \quad \text{implies } T(r) &\sim \lambda^{-1} r^\lambda L(r) && \textit{if } 0 < \lambda \leqq \tfrac{1}{2}, \\ &\sim \lambda^{-1}(\csc \pi\lambda) r^\lambda L(r) && \textit{if } \tfrac{1}{2} \leqq \lambda < 1, \end{aligned}$$

may be found in [4, Corollary 2.1].

(4) Our Theorem 1 asserts that

$$\begin{aligned} T(r) \sim r^\lambda L(r) \quad \text{implies } N(r) &\sim r^\lambda L(r) && \text{if } 0 < \lambda \leqq \tfrac{1}{2}, \\ &\sim r^\lambda L(r) \sin \pi\lambda && \text{if } \tfrac{1}{2} \leqq \lambda < 1. \end{aligned}$$

(5) It was shown by Ahlfors and by Shimizu [8, p. 13] that

$$T(r) = \int_0^r A(t)t^{-1}\,dt + 0(1) \qquad (r \to \infty).$$

Since A is increasing, the methods used to prove the statements in (2) can be applied without change, and we obtain

$$A(r) \sim r^\lambda L(r) \quad \text{iff } T(r) \sim \lambda^{-1} r^\lambda L(r).$$

From the diagram, it follows that regular variation of any one of the functionals implies regular variation of all the rest. The corollary's assertions about ratios follows from the several statements written above.

7. **The cases $\lambda = 0$ and $\lambda = 1$.**

PROPOSITION 0. *Suppose that f has the form* (1.1), *and its order is zero. Then*

(a) *If n or A varies slowly, then so do N, T, and* $\log f(r)$.

(b) *If any one of N,* $\log f(r)$, *or T varies slowly, then so do the other two, and in fact*

$$N(r) \sim T(r) \sim \log f(r) \qquad (r \to \infty).$$

(c) *Slow variation of N does not imply slow variation of n.*

Proof of (b). From Valiron's formula (1.3) we deduce

$$\log f(r) = \int_0^\infty \frac{rN(t)}{(t+r)^2}\,dt,$$

so

$$\log f(r) = N \# J(r) \tag{7.1}$$

with $J(t) = t/(1+t)^2$.

If N is assumed to vary slowly then (7.1) and Lemma 1 yield

$$\frac{\log f(r)}{N(r)} = \frac{(N\cdot 1) \# J(r)}{N(r)} = 1 \# J(r) + o(1) = 1 + o(1) \qquad (r \to \infty). \tag{7.2}$$

Now assume that $\log f$ varies slowly. Since $g(r) = N(r)/\log f(r)$ is bounded, by (3.11), we can apply Lemma 1 to (7.1) again, obtaining

$$1 = (g \cdot \log f) \# J(r)/\log f(r) = g \# J(r) + o(1) \qquad (r \to \infty).$$

Now g is slowly decreasing (by Lemma 3), and

$$\hat{J}(x) = \int_0^\infty J(t)t^{-ix}t^{-1}\,dt = \frac{\pi x}{\sinh \pi x} \neq 0 \qquad (-\infty < x < \infty)$$

so, by Wiener's theorem

$$\lim_{r\to\infty} \frac{N(r)}{\log f(r)} = \lim_{r\to\infty} g(r) = 1. \tag{7.3}$$

Thus, if N (or $\log f$) varies slowly, then by (7.2) (or (7.3)) and the inequality $N(r) \leqq T(r) \leqq \log f(r)$ we see that $N(r) \sim T(r) \sim \log f(r)$.

If T varies slowly, then $T(r) \sim N(r)$ by Theorem 1, and then (7.2) shows that $N(r) \sim \log f(r)$.

Proof of (a). If n (or A) varies slowly, then so does N (or T). See [6, p. 272]. The rest of (a) now follows from (b).

REMARK. If n varies slowly, then $\lim_{r\to\infty} (N(r)/n(r)) = \infty$, since

$$\frac{n(r)}{N(r)} \leqq \frac{N(er)-N(r)}{N(r)} = \frac{N(er)}{N(r)} - 1 = o(1).$$

Proof of (c). Put

$$\begin{aligned} p(t) &= 2^k, \quad \text{if } 2^k \leqq t < 2^{k+1} \qquad (k = 1, 2, \ldots), \\ &= 0, \quad \text{if } -\infty < t < 2. \end{aligned}$$

Then

$$\limsup_{t\to\infty} \frac{p(t+1)}{p(t)} = 2,$$

hence the function with $n(r) = p(\log r)$ does not vary slowly.

On the other hand, $P(x) = \int_{-\infty}^x p(t)\,dt$ is nondecreasing and

$$\frac{P(x+1)}{P(x)} = 1 + \left[\int_x^{x+1} p(t)\,dt\right] \Big/ P(x) = 1 + o(1) \qquad (x \to \infty)$$

which shows that $N(r) = P(\log r)$ does vary slowly.

PROPOSITION 1. *Suppose that f has the form* (1.1) *and its order is* 1. *Then*

(a) *Regular variation of n* (*or N*) *implies that of N* (*or n*), $\log f$, *T, and A.*

(b) *If any one of* $\log f$, *T, or A varies regularly, then so do the other two.*

(c) *Regular variation of T does not imply regular variation of N.*

In the proof, L always stands for a slowly varying function, and "$\sim$" means asymptotic equivalence as the independent variable tends to infinity. Here "regular variation" is always of order 1.

Proof of (a). Statement (2) of §6 still holds, i.e. $n(r) \sim rL(r)$ iff $N(r) \sim rL(r)$.

Consider again $Q(r) = \int_r^\infty (N(t)/t^2)\,dt$.

If N varies regularly, then Q varies slowly [6, p. 272]. Using

$$r^{-1}F(r) = Q \# H_1(r) \tag{7.4}$$

(see the end of §5) and an argument like the one in the proof of Proposition 0(b), we deduce that $\pi r^{-1}F(r) \sim Q(r)$.

Since $\pi F(r)=\int_0^r \log f(t)t^{-1}\,dt$ with $\log f$ increasing, it follows from Landau's theorem that

$$\log f(r) \sim \pi F(r) \sim rQ(r),$$

so that regular variation of N implies that of $\log f$.

To show that slow variation of Q implies regular variation of T, we integrate (3.5) by parts. This gives

$$r^{-1}\,\mathrm{Im}\,F(re^{i\theta}) = Q \# \tilde{K}_\theta(r) \qquad (0 < \theta < \pi)$$

with

$$\tilde{K}_\theta(t) = \frac{-\sin\theta}{\pi}\, t\frac{d}{dt}\frac{1}{t^2+1+2t\cos\theta}.$$

Lemma 1 shows that

$$\mathrm{Im}\,F(re^{i\theta})/rQ(r) = 1 \# \tilde{K}_\theta(r)+o(1) = (\sin\theta)/\pi+o(1)$$

for each $\theta \in (0, \pi)$. Since $T(r)=\max \mathrm{Im}\,F(re^{i\theta})$, it follows that

$$T(r) \sim \mathrm{Im}\,F(re^{i\pi/2}) \sim rQ(r)/\pi.$$

Finally, as in (5) of §6, the regular variation of A follows from that of T.

Proof of (b). Statement (5) of §6 still holds, i.e., $A(r)\sim rL(r)$ iff $T(r)\sim rL(r)$.

Moreover, if T varies regularly, we saw in the proof of Theorem 1 (case $\lambda=1$) that Q varies slowly, which implies, as in part (a), the regular variation of $\log f$.

If $\log f(r)\sim rL(r)$, then $\pi F(r)\sim rL(r)$, and we deduce from (7.4) via Lemma 1 and Wiener's Theorem that $Q(r)\sim L(r)$, which implies, as in (a), the regular variation of T.

Proof of (c). By the preceding remarks, it suffices to find a function for which $n(r)$ does not vary regularly, but $Q(r)$ does vary slowly.

Put

$$\begin{aligned}\phi(x) &= (x-2^{k-1})^{-2} && \text{if } 2^k \leqq x < 2^{k+1},\ k = 2, 3, \dots,\\ &= 0 && \text{if } -\infty < x < 4.\end{aligned}$$

It is not hard to verify that

$$x^{-2} \leqq \phi(x) \leqq 4x^{-2} \qquad (x \geqq 4), \tag{7.5}$$

$$\limsup_{x\to\infty} \frac{\phi(x+1)}{\phi(x)} = \frac{9}{4} \tag{7.6}$$

and

$$x\phi(\log x) \quad \text{is an increasing function of } x. \tag{7.7}$$

By (7.7), we can take

$$n(r) = n(r, 0; f) = [r\phi(\log r)] \qquad ([x] = \text{largest integer} \leqq x).$$

By (7.5), n satisfies

$$\tfrac{1}{2}t(\log t)^{-2} \leqq n(t) \leqq 4t(\log t)^{-2} \qquad (t > t_0) \tag{7.8}$$

and so the entire function f has order 1, genus 0. By (7.6), n does not vary regularly.

Write

$$Q(r) = \int_r^\infty \frac{n(t)}{t^2}\,dt + r^{-1}N(r) = Q_1(r) + r^{-1}N(r).$$

It follows from (7.8) that, as $r \to \infty$,

$$r^{-1}N(r) = O(\log r)^{-2}, \qquad (\log r)^{-1} = O(Q_1(r))$$

and

$$Q_1(er)/Q_1(r) = 1 - \left[\int_r^{er} t^{-2}n(t)\,dt\right] \Big/ Q_1(r) \to 1.$$

Since Q_1 is decreasing, the last statement shows that Q_1 varies slowly, while the other statements show that $r^{-1}N(r) = o(Q_1(r))$. These last two facts imply the slow variation of Q.

Edrei and Fuchs show in [5] that, if f is any entire function, then

$$\lim_{r\to\infty} \frac{N(r, 0)}{T(r)} = 0 \tag{7.9}$$

implies that T varies regularly with order q, where $q \geqq 1$ is an integer. Our example above, together with Theorem 1, shows that (7.9) does not imply regular variation of N.

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